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Linearized Einstein Equations around the Schwarzschild black hole

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Contents

Introduction

In their original work, Regge and Wheeler (1957) studied perturbations of the metric around a Schwarzschild black hole [2]. Roughly speaking, this is done by introducing a perturbation $h_{\mu\nu}$ to the metric $g_{\mu\nu}^{background}$ such that the new metric is:

$$g_{\mu\nu} = g_{\mu\nu}^{background} + h_{\mu\nu} \tag{1}$$

where $|h_{\mu\nu}| \ll 1$. Thus, only terms linear in $h_{\mu\nu}$ are retained in the calculations. Eventually, we will show that a wave equation (named after Regge and Wheeler) with an effective potential governs linear perturbations of a Schwarzschild black hole.

Appropriate Metric for Axisymmetric Spacetimes

Our principle concern is to be able to treat, in full generality, perturbations of axisymmetric spacetimes. In order to do this, we must first find the metric in its most general form given the conditions on our spacetime.

We start by taking two of our coordinates as the time t$(= x^0)$ and the azimuthal angle ϕ$(= x^1)$ about the axis of symmetry. We study the case of space-times that retain their axisymmetry at all times. This restrictions on our space-time requires that the coefficients of the contravariant form of the metric

$$\boldsymbol{g} = g^{ij}\partial_i\partial_j \tag{2}$$

be independent of ϕ [1]. Our first goal is to show that the 3x3-matrix $[g^{ij}]$ (where i,j = 0,2,3) can be made into diagonal form by local coordinate transformation. This is done through the application of the Cotton-Darboux theorem.

Cotton-Darboux Theorem

Theorem (Cotton-Darboux): the metric

$$\boldsymbol{g} = g^{ij}\partial_i\partial_j$$

in three dimensional space (x^0, x^1, x^2) can always be brought to a diagonal form by a local coordinate-transformation.

Proof (taken from Chandrasekhar (1983) [1]):

Definition: a geodesic system of coordinates is constructed by considering a surface $f(x^0, x^1, x^2)$ such that $g^{ij}\partial_i f \partial_j f \neq 0$, letting the geodesics, normal to $f = 0$, be the coordinate lines x^0, and choosing the coordinates $x^1 and\ x^2$ on the surfaces geodesically parallel to f.

Now observe that by choice of geodesic system of coordinates, the metric can be brought to the form:

$$g = e^0\partial_0 + g^{\alpha\beta}\partial_\alpha\partial_\beta \ (e^0 = \pm 1 \ and \ \alpha, \beta \in \{1,2\}) \tag{3}$$

Consider the coordinate transformation:

$$x^{i'} = \phi^{i'}(x^0, x^1, x^2) \ where \ i' \in \{0,1,2\}$$

Where $\phi^{i'}$ are functions of x^0, x^1, x^2 which we try to solve by condition that the metric (3) in the new coordinate system is diagonal. This condition amounts to:

$$g^{i'j'} = e^0 \frac{\partial\phi^{i'}}{\partial x^0}\frac{\partial\phi^{j'}}{\partial x^0} + g^{\alpha\beta}\frac{\partial\phi^{i'}}{\partial x^\alpha}\frac{\partial\phi^{j'}}{\partial x^\beta} = 0 \ where \ (i', j') = (0,1), (1,2), (2,1) \tag{4}$$

Or explicitly:

$$\begin{aligned} e^0\frac{\partial\phi^1}{\partial x^0}\frac{\partial\phi^2}{\partial x^0} &= -g^{\alpha\beta}\frac{\partial\phi^1}{\partial x^\alpha}\frac{\partial\phi^2}{\partial x^\beta} =: K^0 \\ e^0\frac{\partial\phi^2}{\partial x^0}\frac{\partial\phi^0}{\partial x^0} &= -g^{\alpha\beta}\frac{\partial\phi^2}{\partial x^\alpha}\frac{\partial\phi^0}{\partial x^\beta} =: K^1 \\ e^0\frac{\partial\phi^0}{\partial x^0}\frac{\partial\phi^1}{\partial x^0} &= -g^{\alpha\beta}\frac{\partial\phi^0}{\partial x^\alpha}\frac{\partial\phi^1}{\partial x^\beta} =: K^2 \end{aligned} \tag{5}$$

So,

$$\begin{aligned} \frac{\partial\phi^0}{\partial x^0} &= \left(\frac{K^1K^2}{K^0}\right)^{\frac{1}{2}} \\ \frac{\partial\phi^1}{\partial x^0} &= \left(\frac{K^2K^0}{K^1}\right)^{\frac{1}{2}} \\ \frac{\partial\phi^2}{\partial x^0} &= \left(\frac{K^0K^1}{K^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Now suppose that $\phi^0, \phi^1, \phi^2$are functions of two variables $x^1 and\ x^2$, and are specified on the surface (say $x^0 = 0$); and that $K^0, K^1, and\ K^2$ are smooth and nowhere zero on the surface. Then by the Cauchy-Kowalewski theorem, there exists unique functions ϕ^0, ϕ^1, ϕ^2 which satisfy the system of equations (5) and which reduce to the values specified on the surface. The existence of a local coordinate transformation which will bring the metric to the diagonal form is thus established.

Thus, applying the Cotton-Darboux theorem to coordinates (x^0, x^2, x^3) and assuming the desirable coordinate transformation, we find that:

$$g^{02} = g^{03} = g^{23} = 0 \tag{7}$$

We don't have to list all the coefficients due to the symmetry of the metric. We will write the remaining coefficients to in the form:

$$g^{00} = e^{-2\nu}, g^{22} = -e^{-2\mu_2}, g^{33} = -e^{-2\mu_3} \tag{8}$$

$$g^{01} = q_1 e^{-2\nu}, g^{12} = -q_2 e^{-2\mu_2}, g^{13} = -q_3 e^{-2\mu_3} \tag{9}$$

$$g^{11} = q_1^2 e^{-2\nu} - e^{-2\psi} - q_2^2 e^{-2\mu_2} - q_3^2 e^{-2\mu_3} \tag{10}$$

where $\nu, \mu_2, \mu_3, q_1, \psi, q_2, q_3$ are functions of x^0, x^2, and x^3 [1]. Thus the covariant form the metric obtained by lowering indices and writing each term of the metric out is of the form

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\phi - q_1 dt - q_2 dx^2 - q_3 dx^3)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2 \tag{11}$$

Now we have obtained the general metric that we were looking for.

Perturbations of the Schwarzschild Black-Hole

Ricci Tensors for our metric

Our goal is to obtain the relevant perturbation equations by linearizing the field equations about the Schwarzschild solution. The following Ricci tensors of our metric (11) will be useful in our endeavors. In what follows, we use the notation that a comma signifies ordinary partial differentiation. For example: $f_{,i} = \partial_i f$.

These tensors are calculated by Chandrasekhar (1983) and are found to be [1]:

$$-R_{12} = \frac{1}{2} e^{-2\psi-\nu-\mu_3}\left[\left(e^{-3\psi+\nu-\mu_2-\mu_3} Q_{32}\right)_{,3} - \left(e^{-3\psi-\nu-\mu_2+\mu_3} Q_{02}\right)_{,0}\right] \tag{12}$$

$$-R_{13} = \frac{1}{2} e^{-2\psi-\nu-\mu_2}\left[\left(e^{-3\psi+\nu-\mu_2-\mu_3} Q_{23}\right)_{,2} - \left(e^{-3\psi-\nu+\mu_2-\mu_3} Q_{03}\right)_{,0}\right] \tag{13}$$

Where

$$Q_{AB} = q_{A,B} - q_{B,A} \; and \; Q_{A0} = q_{A,0} - q_{1,A} \quad (Where \; A, B \in \{2,3\}) \tag{14}$$

Metric Perturbations

In the unperturbed case, the metric corresponds to the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \tag{15}$$

So we associate x^2, x^3 with r,θ respectively and the metric coefficients are:

$$e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r} = \frac{\Delta}{r^2} \; where \; \Delta = r^2 - 2Mr, e^{2\mu_3} = r, e^{\psi} = rsin\theta \tag{16}$$

$$\omega = q_2 = q_3 = 0 \tag{17}$$

When the metric (14) is perturbed by an external agent and only linear terms are kept:

- q_1, q_2, q_3 become first order quantities
- The functions $\nu, \mu_2, \mu_3, and \; \psi$ experience linear increments $\delta\nu, \delta\mu_2, \delta\mu_3, \delta\psi$respectively

It's easy to see that a perturbation leading to non-vanishing values of q_1, q_2, q_3 is a different kind of perturbation than one leading to increments in $\nu, \mu_2, \mu_3, and \; \psi$. More specifically, the first kind induces a rotation of the black hole while the second kind does not because it is independent of the sign of ϕ [2]. The first kind of perturbation will be referred to as axial, and the second kind will be referred to as polar [1]. As it turns out, these two kinds of permutations decouple in the sense that they can be considered independently of each other. Our interest in the Regge-Wheeler equation leads us to study the axial perturbations.

Axial Perturbation of the Metric

The equations governing non-vanishing q_1, q_2, q_3 are given by

$$R_{12} = R_{13} = 0 \tag{18}$$

From (12,13) we see that we could plug in $\nu, \mu_2, \mu_3, and \; \psi$ into $\delta R_{12} = 0$ and $\delta R_{13} = 0$ we get:

$$\left(e^{3\psi+\nu-\mu_2-\mu_3} Q_{23}\right)_{,3} = -e^{3\psi-\nu-\mu_2+\mu_3} Q_{02,0} \tag{19}$$

$$\left(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23}\right)_{,2} = -e^{3\psi-\nu+\mu_2-\mu_3}Q_{03\,,0} \tag{20}$$

Let

$$Q(t,r,\theta) \coloneqq \Delta Q_{23}\sin^3\theta = \Delta(q_{2\,,3} - q_{3\,,2})\sin^3\theta$$

and substituting for $\nu, \mu_2, \mu_3, and\ \psi$ their unperturbed values (16), we get:

$$\frac{1}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = -\left(q_{1\,,2} - q_{2\,,0}\right)_{,0} \tag{21}$$

$$\frac{\Delta}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = \left(q_{1\,,3} - q_{3\,,0}\right)_{,0} \tag{22}$$

Further, we assume harmonic time dependence of the perturbation $e^{i\omega t}$ where ω is generally a real constant [2]. This corresponds to a Fourier component with frequency –ω when conducting a Fourier analysis of the perturbation [1]. From this time dependence, we can re-write equations (21) and (22) as [1]:

$$\frac{1}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = -i\omega q_{1\,,2} - \omega^2 q_2 \tag{23}$$

$$\frac{\Delta}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta} = i\omega q_{1\,,3} + \omega^2 q_3 \tag{24}$$

Eliminating q_1 from the previous two equations, we obtain:

$$r^4\frac{\partial}{\partial r}\left(\frac{\Delta}{r^4}\frac{\partial Q}{\partial r}\right) + \sin^3\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial\theta}\right) + \frac{\omega^2 r^4}{\Delta}Q = 0 \tag{25}$$

The variables r and θ in (25) can be separated by the substitution [1]:

$$Q(r,\theta) = Q(r)C_{l+2}^{-3/2}(\theta) \tag{26}$$

Where C_n^{ν} denotes the Gegenbauer function governed by:

$$C_{l+2}^{-\frac{3}{2}}(\theta) = \sin^3\theta\frac{d}{d\theta}\frac{1}{sin\theta}\frac{dP_l(\theta)}{d\theta} \tag{27}$$

or

$$C_{l+2}^{-\frac{3}{2}}(\theta) = \left(P_{l\,,\theta\,,\theta} - P_{l\,,\theta}cot\theta\right)\sin^2\theta \tag{28}$$

Where P_l are the Legendre polynomials. The substitution (26) into (25) we get the radial equation [1]:

$$\Delta\frac{d}{dr}\left(\frac{\Delta}{r^4}\frac{dQ}{dr}\right) - (l-1)(l+2)\frac{\Delta}{r^4}Q + \omega^2 Q = 0 \tag{29}$$

Changing to the tortoise coordinates [2]:

$$r_* = r + 2M ln\left(\frac{r}{2M} - 1\right) \tag{30}$$

which satisfies

$$\frac{d}{dr_*} = \left(1 - \frac{2M}{r}\right)\frac{d}{dr} = \frac{\Delta}{r^2}\frac{d}{dr}. \tag{31}$$

Letting:

$$Q(r) = rZ, V = \frac{\Delta}{r^5}\left[\left((l-1)(l+2)+2\right)r - 6M\right] \tag{32}$$

We find that Z satisfies the Schrodinger wave equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)Z = VZ \tag{33}$$

This is the Regge-Wheeler equation that governs the axial perturbations of Schwarzschild black holes. The potential V is referred to as the Regge-Wheeler potential. This potential has a maximum just outside the event horizon r ~ 3.3M as shown in Figure 1 below [3].

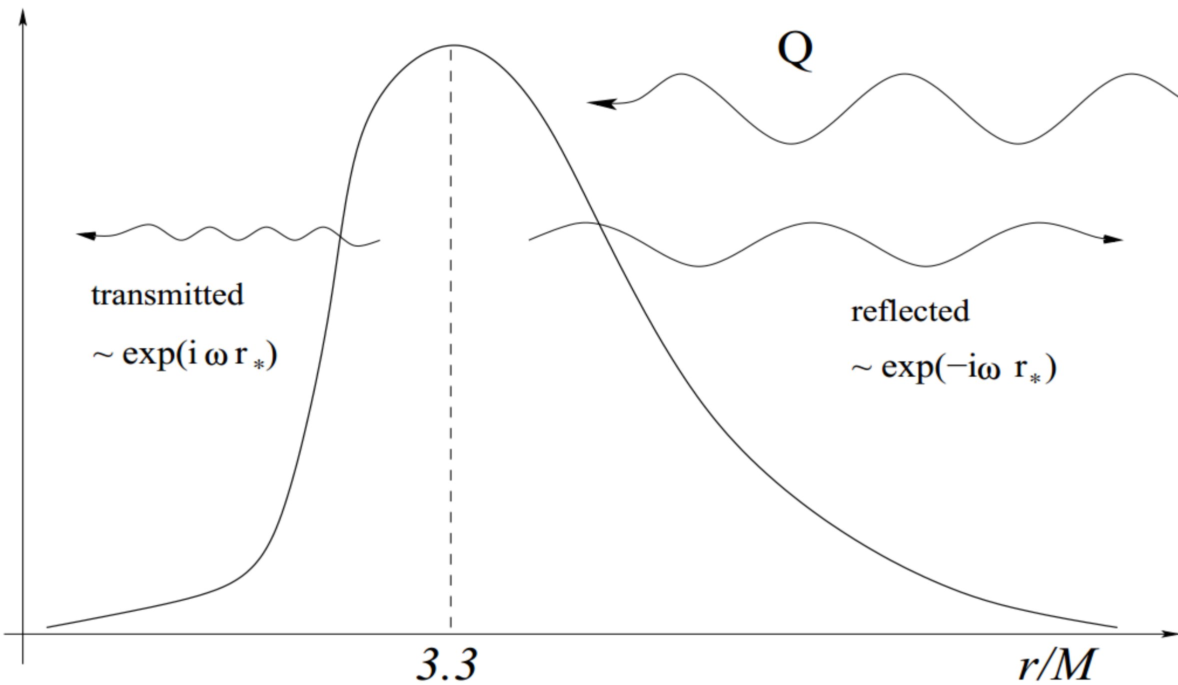

Figure 1 [3]
This graph shows the Regge-Wheeler potential vs. r/M. The maximum is seen to be around r ~ 3.3M. Because the Regge-Wheeler satisfied the Schrodinger equation, the potential can be thought of as a scattering potential barrier. Q represents an incoming wave packet from positive spatial infinity; the other two wavy lines show the reflected and transmitted portion of the incident wave packet.

Because the Regge-wheeler equation satisfies the Schrodinger equation, we can assert that the Regge-wheeler equation shares the well-known properties of a wave equation. We first assert that the Regge-Wheeler potential V is positive everywhere and smooth. Further, the potential decays at infinity. More specifically [1]:

$$V decays\ as\ \frac{1}{r_*^2}\quad as\ r_* \to\ +\infty \tag{34}$$

$$V\ decays\ as\ e^{\frac{r_*}{2M}}\quad as\ r_* \to\ -\infty. \tag{35}$$

Further, since V decays faster than r_*^{-1}, the asymptotic behavior of Z is given by:

$$Z \to e^{\pm\omega r_*}\ as\ r \to \pm\infty. \tag{36}$$

For real ω, the solution represents ingoing and outgoing waves at $\pm\infty$.

The Regge-Wheeler potential can be thought of as a scattering potential [3] as in quantum mechanics. Figure 1 above depicts an incident wave packet Q originating from positive spatial infinity interacting with the scattering potential barrier (in this case the Regge-Wheeler potential). Some of the wave packet is transmitted and the rest is reflected.

Stability of the Schwarzschild Black Hole

The question we wish to answer in this section is the following: given an initial smooth perturbation with a compact support in the exterior region of the black hole, will it remain bounded at all times as it evolves?

We shall take for granted that the linearized equations of polar perturbations of a Schwarzschild black hole are of the form (33), just with a different effective potential. This was derived by Zerilli and is known as the Zerilli equation [2].

The solution comes from elementary quantum theory. The properties we have explored of the potential imply that it is integrable (since it's smooth and decays fast enough). Theorems from quantum theory guarantee that the wave functions belonging to any observable form a complete set, and that any square integrable function that can describe a state of the system can be expanded in terms of them, and further, that the integral of the absolute square of any state function must remain constant with time [1].

We have seen that perturbations of the Schwarzschild black hole are governed by the 1-dimensional wave equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) Z = VZ \tag{37}$$

With asymptotic behavior

$$Z \to e^{\pm \omega r_*} \, as \, r \to \pm\infty. \tag{38}$$

The solutions $Z(r_*, \omega)$ to equation (36), satisfying (37) provide the basic complete set of wave functions [1]. Therefore any smooth compactly supported exterior perturbation can be expressed as an integral over the $Z(r_*, \omega)$ functions in the form [1]:

$$\psi(r_*, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(\omega, 0) Z(r_*, \omega) d\omega \tag{39}$$

Therefore, from quantum theory, the evolved perturbation is expressed by:

$$\psi(r_*, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(\omega, 0) e^{i\omega t} Z(r_*, \omega) d\omega \tag{40}$$

Thus, from the following:

$$\int_{-\infty}^{+\infty} |\, \psi(r_*, 0)|^2 dr_* = \int_{-\infty}^{+\infty} |\, \hat{\psi}(\omega, 0)|^2 d\omega = \int_{-\infty}^{+\infty} |\, \psi(r_*, t)|^2 dr_* \tag{41}$$

we assert that the perturbation is bounded for all times which implies the stability of the Schwarzschild black hole for smooth compactly supported exterior perturbations.

Summary

We started with the task of finding a suitably general metric. After obtaining this metric, we applied a small linear axial perturbation to it. After doing this, we found the linearized Einstein field equations around a Schwarzchild black hole. The resulting Regge-Wheeler equation turns out to satisfy the Schrodinger equation and therefore has nice properties that we know from the study of solutions to the Schrodinger equation. Finally, smooth compactly supported exterior perturbations will remain bounded, therefore implying that Schwarzschild black holes are stable under such perturbations.

References

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[2] V. Frolov, I. Novikov. Black Hole Physics: Basic Concepts and New Developments

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